Energy conservation equation for a radiating pointlike charge in the context of the Abraham-Lorentz versus the Abraham-Becker radiation-reaction force

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With reference to a radiating pointlike charge, the energy conservation equation comprising the effect of the Abraham-Lorentz radiation-reaction force is contrasted with the incorrect energy conservation equation obtained by Hartemann and Luhmann [Phys. Rev. Lett. **74**, 1107 (1995)] on considering instead the Abraham-Becker force that accounts only for a part of the instantaneous radiation-reaction force.
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The issue of the back reaction of radiation on the radiating source has long been an outstanding problem of classical electrodynamics [1–5]. Generally speaking, a radiative recoil force, attributable to the radiating source's own electromagnetic field, is expected on the basis of the conservation of momentum, by virtue of which the rate of electromagnetic momentum loss should just equal the radiation-reaction force. The latter can be obtained from the self-field of the charge in either the Liénard-Wiechert or Heaviside-Feynman form, on making a near-field approximation based on an expansion of the relevant retarded quantities about the iterative solution for retarded time. The expression of the *instantaneous* radiation-reaction force thus obtained, referred to as the Abraham-Lorentz (AL) force, can be expressed in three-vector form as [5]

$$\mathbf{f}^{(\mathrm{AL})}\equiv\frac{2q^2}{3c^3}\,\gamma^2\left\{\ddot{\mathbf{u}}+\frac{\gamma^2}{c^2}\left((\mathbf{u}\cdot\ddot{\mathbf{u}})\mathbf{u}+3(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}+3\,\frac{\gamma^2}{c^2}\,(\mathbf{u}\cdot\dot{\mathbf{u}})^2\mathbf{u}\right)\right\},\tag{1}$$

where $\mathbf{u}=\mathbf{u}(t)$ is the velocity of the charge q at the observer's time t, the dots denote time derivatives, and $\gamma\equiv(1-u^2/c^2)^{-1/2}$. Thus, in particular, the AL force depends on the second time derivative of the velocity of the charge.

Recently, the issue of the radiation reaction on an accelerated charge has been reconsidered [6], and the radiation-reaction force in Abraham-Becker (AB) form, namely,

$$\mathbf{f}^{(\mathrm{AB})}\equiv-\frac{2q^2}{3c^5}\,\gamma^4\left(|\dot{\mathbf{u}}|^2+\frac{\gamma^2}{c^2}\,(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right)\mathbf{u}=-\frac{2q^2}{3c^5}\,\gamma^6\left(|\dot{\mathbf{u}}|^2-\frac{|\mathbf{u}\times\dot{\mathbf{u}}|^2}{c^2}\right)\mathbf{u}\tag{2}$$

has been rederived, which depends on both the velocity and acceleration, but is independent of the second time derivative of the velocity of the charge, in contrast to the AL force, Eq. (1). In Ref. [6] the AB force itself was incorrectly identified as the instantaneous radiation-reaction force, which has led to an erroneous energy conservation equation for the radiating charge. More specifically, the authors of Ref. [6] evaluated the pressure of the radiation on a sphere, the center of which coincides with the position of the radiating charge, and then let the radius of the sphere tend to zero. To assess such a procedure, let us consider the equation for the conservation of the electromagnetic momentum, integrated over a volume within a closed large surface Σ,

$$\frac{\partial}{\partial t}\int d^3r\,\frac{\mathbf{S}(\mathbf{r},t)}{c^2}+\oint_\Sigma d\Sigma\cdot\mathsf{T}(\mathbf{r},t)=-\int d^3r\left(\rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)+\frac{1}{c}\,\mathbf{j}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\right),\tag{3}$$

where $\mathbf{S}/c^2\equiv(1/4\pi c)\mathbf{E}\times\mathbf{B}$ is the electromagnetic momentum density and $\mathsf{T}\equiv(1/4\pi)[(|\mathbf{E}|^2+|\mathbf{B}|^2)\mathsf{I}/2-(\mathbf{EE}+\mathbf{BB})]$ is the Maxwell stress tensor, the part proportional to the unit tensor I being identified as the electromagnetic pressure [2,3]. In the specific case of a point charge, the part proportional to the dyadic of the fields gives no contribution to the reaction force, because the corresponding flux is zero for both the far field, which is transverse with respect to the direction of propagation, and the near field, whose contribution vanishes at large distances from the charge [2]. For the problem at hand, the right-hand side of Eq. (3) is the (instantaneous) force due to the self-electromagnetic field acting on the source charge and current density ρ and $\mathbf{j}$, respectively.

It is apparent from Eq. (3) that the instantaneous momentum balance requires both terms on the left-hand side of Eq. (3) to be accounted for. To single out just the term connected with the radiation pressure, as is done in Ref. [6], amounts to disregarding the time-derivative term in Eq. (3), which is correct, in general, only on the average with respect to time, in which case Eq. (3) reduces, for a point charge, to

$$\mathbf{F}=-\frac{1}{T}\int_{-T/2}^{T/2}dt\,\oint_\Sigma d\Sigma\cdot\mathsf{T}(\mathbf{r},t),\tag{4a}$$

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where

$$\mathbf{F}\equiv\frac{1}{T}\int_{-T/2}^{T/2}dt\int d^3r\left(\rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)+\frac{1}{c}\,\mathbf{j}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\right) \tag{4b}$$

is the *time-averaged* radiation-reaction force. In Eqs. (4), T is an arbitrarily long time, and the corresponding time average is such that it yields zero for the time-derivative term in Eq. (3), that is trivially the case of, e.g., periodic fields. On the basis of Eqs. (4) one cannot, as a rule, identify the instantaneous radiation-reaction force as the integrand of the t integration, as is done in Ref. [6], since such an identification is subject to the indeterminacy of the time derivative of a quantity whose time average is zero. For the specific case of a point charge, on proceeding either from Eq. (4a), as in Ref. [6], or somewhat more directly from Eq. (4b) by means of the Fourier transforms [7], one obtains

$$\mathbf{F}=\frac{1}{T}\int_{-T/2}^{T/2}dt\;\mathbf{f}^{(\mathrm{AB})}(t), \tag{5}$$

where $\mathbf{f}^{(\mathrm{AB})}$ is just the AB force, Eq. (2), which thus emerges in connection with the time-averaged radiation-reaction force.

On the other hand, on comparing the AB force, Eq. (2), with the AL force, Eq. (1), one finds, as expected, that the two differ by a time-derivative-type term, namely,

$$\mathbf{f}^{(\mathrm{AL})}=\frac{d\mathbf{g}}{dt}+\mathbf{f}^{(\mathrm{AB})}, \tag{6}$$

with

$$\mathbf{g}\equiv\frac{2q^2}{3c^3}\,\gamma^2\left(\dot{\mathbf{u}}+\frac{\gamma^2}{c^2}\,(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}\right). \tag{7}$$

Thus the (true) instantaneous radiation-reaction force, i.e., the AL force, comprises two parts, $(d\mathbf{g}/dt)$, that accounts for the rate at which momentum is stored in the self-field nearby the charge, and the AB force, which is connected with the momentum effectively radiated [5]. In particular, for a hyperbolic motion, for which $\ddot{\mathbf{u}}+3(\gamma/c)^2(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}=0$, so that $\mathbf{f}^{(\mathrm{AL})}=0$, $\mathbf{f}^{(\mathrm{AB})}=-(d\mathbf{g}/dt)$, i.e., the AB force just balances the loss of momentum stored in the near field.

On making use of Eq. (1), the rate work done by the charge against the radiation-reaction force is

$$\mathbf{u}\cdot\mathbf{f}^{(\mathrm{AL})}=\frac{dQ}{dt}-P(t), \tag{8a}$$

where

$$Q\equiv\frac{2q^2}{3c^3}\,\gamma^4(\mathbf{u}\cdot\dot{\mathbf{u}}) \tag{8b}$$

is referred to as the Schott energy [5], and

$$P(t)\equiv\frac{2q^2}{3c^3}\,\gamma^4\left(|\dot{\mathbf{u}}|^2+\frac{\gamma^2}{c^2}\,(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right)=\frac{2q^2}{3c^3}\,\gamma^6\left(|\dot{\mathbf{u}}|^2-\frac{|\mathbf{u}\times\dot{\mathbf{u}}|^2}{c^2}\right) \tag{8c}$$

is the power effectively radiated, in accordance with the Liénard formula [2,3]. The time derivative of the Schott energy in Eq. (8a) is identified as the rate energy stored in the near field of the charge [4,5]. For the particular case of hyperbolic motion, for which $\mathbf{f}^{(\mathrm{AL})}=0$, Eq. (8a) requires that $P(t)=dQ/dt$, i.e., that power is effectively radiated at the expense of the energy stored in the near field [4,5].

At this point, it is worth considering the energy conservation equation for the radiating charge. The rate of the energy lost to radiation is

$$\frac{d}{dt}\,(\gamma mc^2)=\mathbf{u}\cdot\mathbf{f}^{(\mathrm{AL})}. \tag{9a}$$

On using Eq. (8a), one can write Eq. (9a) in the form [4]

$$\frac{d}{dt}\,(\gamma mc^2)-\left(\frac{dQ}{dt}-P(t)\right)=0. \tag{9b}$$

Alternatively, on using Eq. (6) and noting that $\mathbf{u}\cdot d\mathbf{g}/dt=-\dot{\mathbf{u}}\cdot\mathbf{g}+d(\mathbf{u}\cdot\mathbf{g})/dt$, with $\mathbf{u}\cdot\mathbf{g}=Q$ and $\dot{\mathbf{u}}\cdot\mathbf{g}=(1/\gamma^2)P(t)$, as follows from Eqs. (7), (8b), and (8c), one can write Eq. (9b) in the form

$$\frac{d}{dt}\,(\gamma mc^2)+\left\{\frac{2q^2}{3c^3}\,\gamma^4\left(|\dot{\mathbf{u}}|^2-\frac{|\mathbf{u}\times\dot{\mathbf{u}}|^2}{c^2}\right)-\frac{dQ}{dt}\right\}-\mathbf{u}\cdot\mathbf{f}^{(\mathrm{AB})}=0. \tag{9c}$$

In contrast with Eq. (9c), the Schott term dQ/dt is missing in the energy conservation equations obtained in Ref. [6], which is a consequence of the incorrect identification of the AB force with the instantaneous radiation-reaction force, in contrast with Eq. (6).

In conclusion, it has been shown that the Abraham-Becker force cannot be identified with the instantaneous radiation-reaction force. Such an incorrect identification was made in Ref. [6], which has led, in particular, to an incorrect energy conservation equation. The true instantaneous radiation-reaction force is the Abraham-Lorentz force, in agreement with the Dirac-Lorentz equation for the motion of a radiating charge [1].

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